Published by Institute of Physics Publishing for SISSA

Received: *February 6, 2006*
Revised: *April 10, 2006*
Accepted: *May 16, 2006*
Published: *June 6, 2006*

Flat coordinates and dilaton fields for three-dimensional conformal sigma models

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Abstract: General forms of the dilaton fields satisfying the vanishing beta function equations of the sigma models in the flat background can be easily expressed in terms of the Riemannian coordinates. Transformations between group coordinates of three-dimensional conformal sigma models in the flat background and their flat, i.e. Riemannian coordinates are found by solving partial differential equations that follow from the transformation properties of the Levi-Civita connection. By the Poisson-Lie transformation we construct dilatons for the dual sigma models. As the Poisson-Lie transformation does not preserve the geometric properties we get dilatons both in the flat and curved backgrounds. The question if the general flat dilatons fulfil restrictions for construction of dual dilatons is investigated.

Keywords: Sigma Models, String Duality.

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1. Introduction

One of the most important and well known aspects of the string theory is its relation to the gravitation. One of the tools for investigating its properties are conformally invariant σ-models in nontrivial backgrounds. They are given by metric, torsion and dilaton field satisfying the so called vanishing β function equations. Their solution is in general a difficult problem and various low dimensional versions are studied to get a better insight into this problem. The main goal of this paper is construction of new three-dimensional σ-models, more precisely, their dilatons. We get them by the Poisson-Lie T-plurality procedure given in [1] and investigate the restrictions following from the requirement that the dilatons do not depend on auxiliary variables appearing in the method.

In the paper [2] we have investigated conformally invariant three-dimensional σ-models on solvable Lie groups that were Poisson-Lie T-dual or plural to σ-models in the flat background with the constant dilaton. Several of them were nontrivial in the sense that they lived in a curved background and had nonvanishing torsion. We have analyzed conditions for construction of dual dilatons by the plurality procedure and found that in some cases we were not able to construct the dual dilaton fields because necessary conditions for application of Poisson-Lie transformation were not satisfied for the constant dilaton of the flat model. In this paper we shall show that these conditions can be satisfied for more general dilatons easily obtainable in terms of the Riemannian coordinates of the flat metrics and we are going to investigate their dual dilatons.

There are two important types of coordinates on the manifolds where the σ-models live. The first one is given by the Lie group structure and follows from the possibility to express the elements of the Lie group (at least in the vicinity of the unit) as a product of elements of one-parametric subgroups. The Poisson-Lie T-dual σ-models are usually expressed in terms of these group coordinates. The other type of coordinates are those in which the metric on the manifold have a special simple form. They are called Riemannian coordinates (see e.g. [3]). The Riemannian coordinates for the flat metrics[1] will be called flat coordinates here. In these coordinates the metric tensors become constant and the Christoffel symbols vanish. The equations of motion (1.2) as well as the vanishing β function equations (1.5)–(1.7) become very simple. That's why it is very desirable to find the transformation between the group and Riemannian coordinates of the σ-models.

In this paper we shall give explicit forms of transformations between these two types of group coordinates, i.e. we are going to express the Riemannian coordinates of the flat metric in parameters of its solvable isometry subgroups. This will enable us to write down the general form of the dilaton field satisfying the vanishing β function equations for the flat model in terms of the group coordinates and consequently the dilaton fields of the dual or plural models in curved backgrounds.

To set our notation let us very briefly review the construction of the Poisson-Lie T-plural σ-models by means of Drinfel'd doubles (For more detailed description see [4], [5], [1], [2]). Principal σ-model can be defined as a field theory on a Lie group G on which a covariant second order tensor field F is given. The action of the σ-model then is

$$S_F[\phi] = \int d^2x \partial_- \phi^i F_{ij}(\phi) \partial_+ \phi^j \tag{1.1}$$

where $\phi: \mathbb{R}^2 \to \mathbb{R}^n$, $n = \dim G$. The equations of motion have the form

$$\partial_- \partial_+ \phi^j + \gamma^j_{rs} \partial_- \phi^r \partial_+ \phi^s = 0, \tag{1.2}$$

where

$$\gamma^j_{rs} = \frac{1}{2} G^{ji} (F_{is,r} + F_{ri,s} - F_{rs,i}) \tag{1.3}$$

and G^{ji} is the inverse of

$$G_{ij} = \frac{1}{2}(F_{ij} + F_{ji}). \tag{1.4}$$

Quantization of the σ-models requires that they be made conformal invariant. This is achieved by addition of another term depending on a scalar (dilaton) field Φ to the action (1.1). To guarantee the conformal invariance of the σ-model (at least at the one-loop level)

[1] More strictly we should speak about pseudometrics as we do not require the positive definiteness.

the fields F and Φ must satisfy the so called vanishing β function equations

$$0 = R_{ij} - \bigtriangledown_i \bigtriangledown_j \Phi - \frac{1}{4} H_{imn} H_j^{mn}, \tag{1.5}$$

$$0 = H_{kij} \bigtriangledown^k \Phi + \bigtriangledown^k H_{kij}, \tag{1.6}$$

$$0 = R - 2 \bigtriangledown_k \bigtriangledown^k \Phi - \bigtriangledown_k \Phi \bigtriangledown^k \Phi - \frac{1}{12} H_{kmn} H^{kmn}, \tag{1.7}$$

where covariant derivatives $\bigtriangledown_k$, Ricci tensor R_{ij} and scalar curvature R are calculated from the (pseudo)metric (1.4) that is also used for lowering and raising indices. The components of torsion are defined as

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}, \tag{1.8}$$

where

$$B_{ij} = \frac{1}{2}(F_{ij} - F_{ji}). \tag{1.9}$$

We shall be interested in σ-models that satisfy the vanishing β function equations and moreover are Poisson-Lie T- dualizable i.e. satisfy [5]

$$\mathcal{L}_{v_i}(F)_{\mu\nu} = F_{\mu\kappa} v_j^\kappa \tilde{f}_i^{jk} v_k^\lambda F_{\lambda\nu}, \qquad i = 1, \ldots, \dim G, \tag{1.10}$$

where v_i form a basis of left-invariant fields on G and $\tilde{f}_i^{jk}$ are structure coefficients of a Lie group $\tilde{G}$ such that $\dim \tilde{G} = \dim G$. If F satisfies the equation (1.10) then the equations of motion of the σ-model can be rewritten (see [4, 5]) as equations for maps to the six-dimensional Drinfel'd double $D = (G|\tilde{G})$ — connected Lie group whose Lie algebra $\mathcal{D}$ admits a decomposition into two subalgebras that are maximally isotropic with respect to a bilinear, symmetric, nondegenerate, ad-invariant form on $\mathcal{D}$. This decomposition $\mathcal{D} = (\mathcal{G}|\tilde{\mathcal{G}})$ is called the Manin triple.

The Lagrangian of dualizable σ-models can be written in terms of right-invariant fields on a Lie group G that is a subgroup of the Drinfel'd double as

$$L = F_{ij}(\phi)\partial_- \phi^i \partial_+ \phi^j = E_{ab}(g)(\partial_- g g^{-1})^a (\partial_+ g g^{-1})^b. \tag{1.11}$$

The functions ϕ are obtained by the composition $\phi^j = y^j \circ g$ of a map $g : \mathbb{R}^2 \to G$ and a coordinate map $y : U_g \to \mathbb{R}^n$ of a neighborhood of an element $g(x_+, x_-) \in G$,

$$E(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g) a(g)^{-1} = -\Pi(g)^t, \tag{1.12}$$

and $a(g), b(g), d(g)$ are submatrices of the adjoint representation of the group G on the Lie algebra of the Drinfel'd double [2]

$$Ad(g)^t = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}. \tag{1.13}$$

The main problem of this paper is solution of the equations (1.5)–(1.7) and we shall use the flat coordinates and the Poisson-Lie transformations of dilatons to solve them.

[2] t denotes transposition.

2. Poisson-Lie transformation

The fact that for a Drinfel'd double several decompositions of its Lie algebra $\mathcal{D}$ into Manin triples $(\mathcal{G}|\tilde{\mathcal{G}})$ may exist leads to the notion of Poisson-Lie T-plurality [1]. Namely, let $\{X_j, \tilde{X}^k\},\ j,k \in \{1,\ldots,n\}$ be generators of Lie subalgebras $\mathcal{G}, \tilde{\mathcal{G}}$ of the Manin triple associated with the Lagrangian (1.11) and $\{U_j, \tilde{U}^k\}$ are generators of some other Manin triple $(\mathcal{G}_U|\tilde{\mathcal{G}}_U)$ of the same Drinfel'd double related by the $2n \times 2n$ transformation matrix as

$$\begin{pmatrix} \vec{X} \\ \vec{\tilde{X}} \end{pmatrix} = \begin{pmatrix} P & T \\ R & S \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{\tilde{U}} \end{pmatrix}, \tag{2.1}$$

where

$$\vec{X} = (X_1, \ldots, X_n)^t, \ldots,\ \vec{\tilde{U}} = (\tilde{U}^1, \ldots, \tilde{U}^n)^t.$$

The transformed model is then given by the Lagrangian of the form (1.11) but with $E(g)$ replaced by

$$\widetilde{E}_U(g_u) = M(N + \Pi_U\, M)^{-1} = (\widetilde{E}_0^{-1} + \Pi_U)^{-1}, \tag{2.2}$$

where

$$M = S^t E_0 - T^t, \qquad N = P^t - R^t E_0, \qquad \widetilde{E}_0 = M N^{-1} \tag{2.3}$$

and Π_U is calculated by (1.12) from the adjoint representation of the group G_U generated by $\{U_j\}$. Note that for $P = S = 0,\ T = R = \mathbf{1}$ we get the dual model with $\widetilde{E}_0 = E_0^{-1}$, corresponding to the interchange $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ so that the duality transformation is a special case of the plurality transformation (2.1) – (2.3).

For the quantum σ-models the Poisson-Lie transformation of the tensor F that follows from (2.2) must be accompanied by the transformation of the dilaton [1]

$$\Phi_U = \Phi + \ln|\mathrm{Det}(N + \Pi_U M)| - \ln|\mathrm{Det}(\mathbf{1} + \Pi E_0)| + \ln|\mathrm{Det}\, a_U| - \ln|\mathrm{Det}\, a| \tag{2.4}$$

where Π_U, a_U, are calculated by (1.12) and (1.13) but from the adjoint representation of the group G_U. The transformed dilaton Φ_U then satisfy the vanishing β function equations if the dilaton Φ does.

Unfortunately, the right-hand side of the formula (2.4) may depend on the coordinates of the auxiliary group $\widetilde{G}$. That's why the transformation of the dilaton field cannot be applied in general but only if the following theorem holds [2]

Theorem 1. *The dilaton (2.4) for the model defined on the group G_U exists if and only if*

$$\tilde{U}\Phi^{(0)}(g.\tilde{g}) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(0)}\left(g.\tilde{g}.\ \exp(t\tilde{U})\right)|_{t=0} = 0,\ \forall g \in G_U,\ \forall \tilde{g} \in \tilde{G}_U,\ \forall \tilde{U} \in \tilde{\mathcal{G}}_U, \tag{2.5}$$

where $\tilde{U} \in \tilde{\mathcal{G}}_U$ is extended as a left-invariant vector field on D and

$$\Phi^{(0)}(g) = \Phi(g) - \ln|\mathrm{Det}(\mathbf{1} + \Pi(g)E_0)| - \ln|\mathrm{Det}\, a(g)|. \tag{2.6}$$

For applications it is much easier to check a weaker necessary condition.

Theorem 2. *A necessary condition for the existence of the dilaton (2.4) for the model defined on the group G_U is*

$$\tilde{U}\Phi^{(0)}(e) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(0)}(\exp(t\tilde{U}))|_{t=0} = 0, \ \forall \tilde{U} \in \tilde{\mathcal{G}}_U, \tag{2.7}$$

where e is the unit of the Drinfel'd double D.

For parametrization of $g \in G$ in the form

$$g(y) = \exp(y_1 X_1)\ \exp(y_2 X_2)\ \exp(y_3 X_3), \tag{2.8}$$

where y_j are coordinates on the group manifold and X_j are generators of the corresponding Lie algebra, the condition (2.7) can be rewritten (see [2]) as

$$R^{jk}\frac{\partial \Phi^{(0)}(y)}{\partial y_j}|_{y=0} = 0, \tag{2.9}$$

where R is the submatrix in (2.1).

For some of the σ-models with constant dilaton field the condition (2.9) could not be satisfied and in those cases we were not able to find the transformed dilaton Φ_U that satisfy the vanishing β function equations. The possibility to find the general dilaton fields for the flat models offers a possibility to overcome this obstacle and obtain more general dilatons in curved backgrounds.

3. Flat models and their Riemannian coordinates

In the paper [2] the semiabelian Drinfel'd doubles $(G|1)$, for which $\mathcal{G}$ in the decomposition $(\mathcal{G}|\tilde{\mathcal{G}})$ are solvable Bianchi algebras $\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6_0}, \mathbf{7_0}$ (see [6, 7]) and $\tilde{\mathcal{G}}$ is the three-dimensional Abelian Lie algebra, were investigated and a classification of conformal invariant Poisson-Lie T-dualizable σ-models with constant dilaton field was done. All the models were torsionless and flat in the sense that their Riemann-Christoffel tensor vanishes.

The flat (pseudo)metrics corresponding to the investigated Drinfel'd doubles expressed in the group coordinates are

$(2|1)$:

$$G(y)_{ij} = \begin{pmatrix} 0 & u & v \\ u & q & g + uy_2 \\ v & g + uy_2 & r + 2vy_2 \end{pmatrix}, \tag{3.1}$$

$(3|1)$:

$$G(y)_{ij} = \begin{pmatrix} p & u + ze^{-2y_1} & -u + ze^{-2y_1} \\ u + ze^{-2y_1} & q & -q \\ -u + ze^{-2y_1} & -q & q \end{pmatrix}, \tag{3.2}$$

$(4|1)$:

$$G(y)_{ij} = \begin{pmatrix} p & (vy_1+u)e^{-y_1} & ve^{-y_1} \\ (vy_1+u)e^{-y_1} & qe^{-2y_1} & 0 \\ ve^{-y_1} & 0 & 0 \end{pmatrix}, \tag{3.3}$$

$(5|1)$:

$$G(y)_{ij} = \begin{pmatrix} p & ue^{-y_1} & ve^{-y_1} \\ ue^{-y_1} & \frac{g^2}{r}e^{-2y_1} & ge^{-2y_1} \\ ve^{-y_1} & ge^{-2y_1} & re^{-2y_1} \end{pmatrix}, \tag{3.4}$$

$(6_0|1)$:

$$G(y)_{ij} = \begin{pmatrix} p & 0 & v+py_2 \\ 0 & -p & g-py_1 \\ v+py_2 & g-py_1 & r+2gy_1+2vy_2+p(y_2^2-y_1^2) \end{pmatrix}, \tag{3.5}$$

$(7_0|1)$:

$$G(y)_{ij} = \begin{pmatrix} p & 0 & z+py_2 \\ 0 & p & g-py_1 \\ z+py_2 & g-py_1 & r-2gy_1+2zy_2+p(y_1^2+y_2^2) \end{pmatrix} \tag{3.6}$$

where u, v, p, q, g, r, z are arbitrary real constants.

Beside these models, solutions of the vanishing β function equations with flat metrics and *nonconstant* dilaton fields Φ were found by the Poisson-Lie T-plurality $(1|6_0) \cong (5ii|6_0) \cong (6_0|1)$. The metrics and the dilaton fields expressed in the group coordinates read

$(1|6_0)$:

$$G(y)_{ij} = K(y_1,y_2)^{-1} \begin{pmatrix} -k^2qy_1{}^2 & k^2qy_1y_2 & -k(1+ky_1) \\ k^2qy_1y_2 & q(-1+k^2y_2{}^2) & k^2y_2 \\ -k(1+ky_1) & k^2y_2 & 0 \end{pmatrix}, \tag{3.7}$$

$$\Phi = \ln|(K(y_1,y_2)| + C, \tag{3.8}$$

where k, q are constants and

$$K(y_1,y_2) = 1 + 2ky_1 + k^2(y_1{}^2 - y_2{}^2).$$

$(5ii|6_0)$:

$$\begin{aligned} G(y)_{11} &= \frac{q\,(w^2-1)}{4\,W(y_1,y_2)}\,(1+e^{2y_1+2y_2}-2\,e^{2y_1+y_2})^2, \\ G(y)_{21} &= \frac{q}{4\,W(y_1,y_2)}\,(1-2e^{2y_1+y_2}+e^{2y_1+y_2})\left(w^2(1-2e^{y_1}+e^{2y_1+2y_2})-1-e^{2y_1+2y_2}\right), \\ G(y)_{22} &= \frac{q}{4\,W(y_1,y_2)}\,\left(w^2(1-2e^{y_1}+e^{2y_1+2y_2})^2-(1+e^{2y_1+2y_2})^2\right), \\ G(y)_{31} &= \frac{w}{2\,W(y_1,y_2)}\,e^{y_1+y_2}\left((2e^{2y_1+y_2}-e^{2y_1+2y_2})(w-1)-w-1\right), \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
G(y)_{32} &= \frac{w}{2\,W(y_1,y_2)} e^{y_1+y_2} \left(2w\, e^{y_1} - e^{2y_1+2y_2}(w-1) - w - 1\right), \\
G(y)_{33} &= 0 \\
\Phi &= \ln\left|(1+w)e^{-(y_1+y_2)} + w(1-2e^{-y_2})\right| + \ln\left|(w-1)e^{y_1+y_2} - w\right| + C,
\end{aligned} \tag{3.10}
$$

where w is a constant and

$$
W(y_1,y_2) = e^{y_1+y_2}((w-1)\, e^{y_1+y_2} - w)(1 + w - 2w\, e^{y_1} + w\, e^{y_1+y_2}).
$$

All the models can also have nonzero antisymmetric part B of the tensor F but the corresponding torsions H_{ijk} given by (1.8) are zero so that we shall assume that $F_{ij} = G_{ij}$ in the following. In spite of the fact that all the metrics above are flat, the task to find coordinates for which the metrics become constant is not trivial.

For finding the flat coordinates we shall use the formula for transformation of the Levi-Civita connection

$$
\Gamma^i_{jk}(y) = \frac{1}{2} G^{li}\Big(\frac{\partial G_{kl}}{\partial y_j} + \frac{\partial G_{jl}}{\partial y_k} - \frac{\partial G_{kj}}{\partial y_l}\Big). \tag{3.11}
$$

that reads as

$$
\Gamma^i_{jk}(y) = \frac{\partial y_i}{\partial \xi^l}\frac{\partial \xi_m}{\partial y_j}\frac{\partial \xi_n}{\partial y_k}\Gamma'^l_{mn}(\xi) + \frac{\partial y_i}{\partial \xi_l}\frac{\partial^2 \xi_l}{\partial y_j \partial y_k}. \tag{3.12}
$$

The components of $\Gamma'^l_{mn}(\xi)$ in the flat coordinates vanish and we get the system of partial differential equations for $\xi(y)$

$$
\frac{\partial^2 \xi_i}{\partial y_j \partial y_k} = \Gamma^l_{jk}\frac{\partial \xi_i}{\partial y_l}. \tag{3.13}
$$

The system is linear and moreover separated with respect to the unknowns ξ_i's. The possibility to solve it explicitly depends on the form of Γ^l_{jk}. We were able to find general explicit solutions for the metrics given above that together with the suitable initial conditions will produce the Riemannian coordinates. The initial condition

$$
\left[\frac{\partial \xi_k}{\partial y_i}\right]_{\vec{y}=\vec{0}} = \delta^i_k \tag{3.14}
$$

produce the coordinates in which the metric acquires the constant form $\tilde{G}(\xi) = G(y=0)$ that can be further diagonalized.

In the following we shall present solution of the equations (3.13) in detail for the metric (3.5) and write down the results for the other metrics. The flat coordinates for the metric (3.1) were used in [8] for solution of equations of motion of a model in curved background. The flat coordinates for the metrics (3.7) and (3.9) produce dilatons that generalize (3.8) and(3.10) and provide an independent check of the formula (2.4). The flat coordinates for the metrics (3.3) and (3.4) will be used for finding nonconstant dilatons in models with curved backgrounds in the next section.

3.1 Solving the equations for flat coordinates of the σ-model on $(6_0|1)$

The nonzero components of the affine connection for the metric (3.5) are

$$\begin{aligned} \Gamma^1_{23} &= 1\,, \qquad \Gamma^1_{33} = \frac{-g+py_1}{p}\,, \\ \Gamma^2_{13} &= 1\,, \qquad \Gamma^2_{33} = \frac{v+py_2}{p}, \end{aligned} \tag{3.15}$$

so that the equations (3.13) read

$$\frac{\partial^2\xi}{\partial y_1\partial y_1} = 0, \tag{3.16}$$

$$\frac{\partial^2\xi}{\partial y_1\partial y_2} = 0, \tag{3.17}$$

$$\frac{\partial^2\xi}{\partial y_1\partial y_3} = \frac{\partial\xi}{\partial y_2}, \tag{3.18}$$

$$\frac{\partial^2\xi}{\partial y_2\partial y_2} = 0, \tag{3.19}$$

$$\frac{\partial^2\xi}{\partial y_2\partial y_3} = \frac{\partial\xi}{\partial y_1}, \tag{3.20}$$

$$\frac{\partial^2\xi}{\partial y_3\partial y_3} = \left(\frac{-g+py_1}{p}\right)\frac{\partial\xi}{\partial y_1} + \left(\frac{v+py_2}{p}\right)\frac{\partial\xi}{\partial y_2}. \tag{3.21}$$

From (3.16) and (3.17) we get

$$\xi = f(y_3)\,y_1 + h(y_2,y_3) \tag{3.22}$$

and the equations (3.19) and (3.18) imply

$$h(y_2,y_3) = f'(y_3)\,y_2 + b(y_3). \tag{3.23}$$

The equation (3.20) gives

$$f(y_3) = ce^{y_3} + de^{-y_3} \tag{3.24}$$

and from (3.21) we get the equation for the function b

$$\frac{d^2b}{dy_3{}^2} = -\frac{g}{p}(ce^{y_3}+de^{-y_3}) + \frac{v}{p}(ce^{y_3}-de^{-y_3})$$

solved by

$$b(y_3) = -\frac{g}{p}\left(ce^{y_3}+de^{-y_3}\right) + \frac{v}{p}\left(ce^{y_3}-de^{-y_3}\right) + my_3 + n. \tag{3.25}$$

The general solution of the system (3.16)–(3.21) then is

$$\xi(y_1,y_2,y_3) = c(y_1+y_2)e^{y_3} + d(y_1-y_2)e^{-y_3} + \frac{c(v-g)}{p}e^{y_3} - \frac{d(v+g)}{p}e^{-y_3} + my_3 + n, \tag{3.26}$$

where m, n, c, d are integration constants. As the transformation formulas (3.13) are the same for all the coordinate components ξ_i we can write the flat coordinates in general as

$$\begin{aligned}
\xi_1(y_1,y_2,y_3) &= c_1(y_1+y_2)e^{y_3} + d_1(y_1-y_2)e^{-y_3} + \frac{c_1(v-g)}{p}e^{y_3} - \frac{d_1(v+g)}{p}e^{-y_3} \\
&\quad + m_1 y_3 + n_1, \\
\xi_2(y_1,y_2,y_3) &= c_2(y_1+y_2)e^{y_3} + d_2(y_1-y_2)e^{-y_3} + \frac{c_2(v-g)}{p}e^{y_3} - \frac{d_2(v+g)}{p}e^{-y_3} \\
&\quad + m_2 y_3 + n_2, \\
\xi_3(y_1,y_2,y_3) &= c_3(y_1+y_2)e^{y_3} + d_3(y_1-y_2)e^{-y_3} + \frac{c_3(v-g)}{p}e^{y_3} - \frac{d_3(v+g)}{p}e^{-y_3} \\
&\quad + m_3 y_3 + n_3.
\end{aligned} \tag{3.27}$$

and the integration constants will be determined by the required form of the constant metric. When we choose

$$\left[\frac{\partial \xi_k}{\partial y_i}\right]_{\vec{y}=\vec{0}} = \delta_i^k$$

then

$$\begin{aligned}
\xi_1(y_1,y_2,y_3) &= y_1\cosh(y_3) + y_2\sinh(y_3) + \frac{v}{p}\sinh(y_3) - \frac{g}{p}\cosh(y_3) - \frac{v}{p}y_3 + n_1, \\
\xi_2(y_1,y_2,y_3) &= y_1\sinh(y_3) + y_2\cosh(y_3) + \frac{v}{p}\cosh(y_3) - \frac{g}{p}\sinh(y_3) + \frac{g}{p}y_3 + n_2, \\
\xi_3(y_1,y_2,y_3) &= y_3 + n_3.
\end{aligned} \tag{3.28}$$

and

$$\tilde{G}(\xi) = \begin{pmatrix} p & 0 & v \\ 0 & -p & g \\ v & g & r \end{pmatrix}. \tag{3.29}$$

This constant form can be transformed by linear transformation

$$\begin{aligned}
y'_1 &= \left(\sqrt{|p|}\right)\xi_1 + \varepsilon\left(\frac{v}{\sqrt{|p|}}\right)\xi_3, \\
y'_2 &= \left(\sqrt{|p|}\right)\xi_2 - \varepsilon\left(\frac{g}{\sqrt{|p|}}\right)\xi_3, \\
y'_3 &= \left(\sqrt{\left|r + \left(\frac{g^2}{p} - \frac{v^2}{p}\right)\right|}\right)\xi_3,
\end{aligned} \tag{3.30}$$

where

$$\varepsilon = \mathrm{sign}(p), \qquad \lambda = \mathrm{sign}\left(r + \frac{g^2}{p} - \frac{v^2}{p}\right)$$

(for $p = 0$ or $r + \frac{g^2}{p} - \frac{v^2}{p} = 0$ the metric is not invertible) to the diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{3.31}$$

Solution of equations (3.13) for the other metrics is a bit more complicated, nevertheless, we were able to find the flat coordinates in all investigated cases. Results are given below.

3.2 Flat coordinates for the σ-model on $(2|1)$

The nonzero components of the affine connection for the metric (3.1) are

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{-u^2g - u^3y_2 + uvq}{u^2r - 2uvg + v^2q}, & \Gamma^1_{23} &= \frac{-vug - u^2vy_2 + v^2q}{u^2r - 2uvg + v^2q}, & \Gamma^1_{33} &= \frac{-vur - uv^2y_2 + v^2g}{u^2r - 2uvg + v^2q},\\
\Gamma^2_{22} &= \frac{-u^2v}{u^2r - 2uvg + v^2q}, & \Gamma^2_{23} &= \frac{-uv^2}{u^2r - 2uvg + v^2q}, & \Gamma^2_{33} &= \frac{-v^3}{u^2r - 2uvg + v^2q},\\
\Gamma^3_{22} &= \frac{u^3}{u^2r - 2uvg + v^2q}, & \Gamma^3_{23} &= \frac{u^2v}{u^2r - 2uvg + v^2q}, & \Gamma^3_{33} &= \frac{uv^2}{u^2r - 2uvg + v^2q}.
\end{aligned}
\tag{3.32}
$$

The general solution of the equations (3.13) is

$$
\begin{aligned}
\xi(y_1, y_2, y_3) = {} & a - 6d(u\rho - v\omega)^2 y_1 + b\,Y + c\,Y^2 + d(u\rho - v\omega)Y^3 + \\
& (2c - 6d\rho\omega)Z + 3duv\,Z^2 - 6dv\omega\,Y\,Z
\end{aligned}
\tag{3.33}
$$

where a, b, c, d are integration constants and

$$Y = u\,y_2 + v\,y_3, \quad Z = \omega\,y_2 + \rho\,y_3,$$

$$\omega = g\,u - q\,v, \quad \rho = r\,u - g\,v.$$

When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$
\begin{aligned}
\xi_1(y_1, y_2, y_3) = {} & \frac{(6y_1u^2r - 12y_1uvg + 6y_1v^2q - 3u^2{y_2}^2g + 3u{y_2}^2vq - u^3{y_2}^3)}{6(u^2r - 2uvg + v^2q)}\\
& + \frac{(-3u^2vy_3{y_2}^2 - 3uy_2v^2{y_3}^2 - 6uy_2vy_3g + 6y_2v^2y_3q - v^3{y_3}^3)}{6(u^2r - 2uvg + v^2q)}\\
& + \frac{(v^2{y_3}^2g - v{y_3}^2ur)}{2(u^2r - 2uvg + v^2q)} + d_1,\\
\xi_2(y_1, y_2, y_3) = {} & \frac{(-u^2v{y_2}^2 - 2uv^2y_3y_2 + 2ru^2y_2 - 4uvy_2g + 2y_2v^2q - v^3{y_3}^2)}{2(u^2r - 2uvg + v^2q)} + d_2,\\
\xi_3(y_1, y_2, y_3) = {} & \frac{(u^3{y_2}^2 + 2u^2vy_3y_2 + uv^2{y_3}^2 + 2y_3u^2r - 4vy_3ug + 2v^2y_3q)}{2(u^2r - 2uvg + v^2q)} + d_3
\end{aligned}
\tag{3.34}
$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} 0 & u & v \\ u & q & g \\ v & g & r \end{pmatrix}. \tag{3.35}$$

By the linear transformation

$$y_1' = \left(\frac{u}{\varepsilon\sqrt{|q|}}\right)\xi_1 + \left(\sqrt{|q|}\right)\xi_2 + \left(\frac{g}{\varepsilon\sqrt{|q|}}\right)\xi_3,$$

$$y_2' = \left(\frac{u}{\sqrt{|q|}}\right)\xi_1 + \left(\left(\frac{ug}{|q|} - \frac{v}{\varepsilon}\right)\left(\frac{\sqrt{|q|}}{u}\right)\right)\xi_3,$$
$$y_3' = \sqrt{\left|\left(\frac{gu}{q} - v\right)^2 \frac{q}{u^2} - \frac{g^2}{q} + r\right|}\,\xi_3, \tag{3.36}$$

where

$$\varepsilon = \text{sign}(q), \qquad \lambda = \text{sign}\left(\left(\frac{gu}{q} - v\right)^2 \frac{q}{u^2} - \frac{g^2}{q} + r\right)$$

we can transform the metric tensor (3.1) to the constant diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{3.37}$$

3.3 Flat coordinates for the σ-model on (3|1)

The nonzero components of the affine connection for the metric (3.2) are

$$\Gamma^1_{11} = -2\,, \qquad \Gamma^2_{11} = \frac{(pq-u^2)e^{2y_1} + zu}{qz}\,, \qquad \Gamma^3_{11} = \frac{(pq-u^2)e^{2y_1} - zu}{qz}\,. \tag{3.38}$$

The general solution of the equations (3.13) is

$$\xi(y_1, y_2, y_3) = cy_3 + ay_2 + \frac{u(a-c)}{2q}y_1 + \frac{(pq-u^2)(a+c)}{8qz}e^{2y_1} + de^{-2y_1} + b, \tag{3.39}$$

where a, b, c, d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\xi_1(y_1, y_2, y_3) = -\frac{1}{2}e^{-2y_1} + b_1,$$
$$\xi_2(y_1, y_2, y_3) = y_2 + \frac{u}{2q}y_1 + \frac{(pq-u^2)}{8qz}e^{2y_1} + \frac{(pq-u^2+2uz)}{8qz}e^{-2y_1}, \tag{3.40}$$
$$+b_2, \tag{3.41}$$
$$\xi_3(y_1, y_2, y_3) = y_3 - \frac{u}{2q}y_1 + \frac{(pq-u^2)}{8qz}e^{2y_1} + \frac{(pq-u^2-2uz)}{8qz}e^{-2y_1}$$
$$+b_3,$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} p & u+z & z-u \\ u+z & q & -q \\ z-u & -q & q \end{pmatrix}. \tag{3.42}$$

By the linear transformation

$$y_1' = \left(\sqrt{|p|}\right)\xi_1 + \left(\frac{u+z}{\varepsilon\sqrt{|p|}}\right)\xi_2 + \left(\frac{z-u}{\varepsilon\sqrt{|p|}}\right)\xi_3,$$

$$y'_2 = \left(\sqrt{\left|q - \frac{(u+z)^2}{p}\right|}\right)\xi_2 - \left(\frac{\delta\left(q + \frac{(z^2-u^2)}{p}\right)}{\sqrt{\left|q - \frac{(u+z)^2}{p}\right|}}\right)\xi_3,$$

$$y'_3 = \left(\sqrt{\left|q - \frac{\left(\frac{z^2-u^2}{p} + q\right)^2}{\left(q - \frac{(u+z)^2}{p}\right)} - \frac{(z-u)^2}{p}\right|}\right)\xi_3, \tag{3.43}$$

where

$$\varepsilon = \mathrm{sign}(p), \quad \delta = \mathrm{sign}\left(q - \frac{(u+z)^2}{p}\right), \quad \lambda = \mathrm{sign}\left(q - \frac{\left(\frac{z^2-u^2}{p} + q\right)^2}{\left(q - \frac{(u+z)^2}{p}\right)} - \frac{(z-u)^2}{p}\right)$$

we can transform the metric tensor (3.2) to the constant diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & -\lambda \end{pmatrix}. \tag{3.44}$$

3.4 Flat coordinates for the σ-model on $(4|1)$

The nonzero components of the affine connection for the metric (3.3) are

$$\begin{aligned}
\Gamma^1_{11} &= -1 & \Gamma^2_{11} &= \frac{v}{q}e^{y_1} & \Gamma^2_{12} &= -1 \\
\Gamma^2_{21} &= -1 & \Gamma^3_{11} &= \left(\frac{p}{v} - \frac{u}{q} - \frac{v}{q}y_1\right)e^{y_1} & \Gamma^3_{12} &= \frac{u}{v} + y_1 \\
\Gamma^3_{22} &= \frac{q}{v}e^{-y_1}\,. & & & &
\end{aligned} \tag{3.45}$$

The general solution of the equations (3.13) is

$$\begin{aligned}
\xi(y_1, y_2, y_3) = {} & cy_3 + \frac{qc}{2v}y_2^2 e^{-y_1} + ay_2 e^{-y_1} + cy_1 y_2 + \frac{cu}{v}y_2 - cy_2 + \frac{av}{q}y_1 \\
& + \frac{pc}{2v}e^{y_1} + \frac{cv}{2q}e^{y_1} + de^{-y_1} + b,
\end{aligned} \tag{3.46}$$

where a, b, c, d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\begin{aligned}
\xi_1(y_1, y_2, y_3) &= -e^{-y_1} \\
\xi_2(y_1, y_2, y_3) &= y_2 e^{-y_1} + \frac{v}{q}y_1 + \frac{v}{q}e^{-y_1} \\
\xi_3(y_1, y_2, y_3) &= \frac{(pq - 2uv)}{2qv}e^{-y_1} + \frac{v}{2q}e^{-y_1} + y_3 + \frac{q}{2v}y_2^2 e^{-y_1} + y_1 y_2 + \frac{u}{v}y_2 \\
&\quad - y_2 + \frac{p}{2v}e^{y_1} - \frac{u}{v}y_2 e^{-y_1} + y_2 e^{-y_1} + \frac{(v-u)}{q}y_1 - \frac{v}{2q}e^{y_1}.
\end{aligned} \tag{3.47}$$

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and

$$\widetilde{G}(\xi) = \begin{pmatrix} p & u & v \\ u & q & 0 \\ v & 0 & 0 \end{pmatrix}. \tag{3.48}$$

By the linear transformation

$$\begin{aligned} y'_1 &= \sqrt{|q|}\xi_2 + \frac{\varepsilon u}{\sqrt{|q|}}\xi_1, \\ y'_2 &= \sqrt{\left|p - \frac{u^2}{q}\right|}\xi_1 + \left(\frac{\delta v}{\sqrt{\left|p - \frac{u^2}{q}\right|}}\right)\xi_3, \\ y'_3 &= \left(\sqrt{\left|\frac{v^2}{p - \frac{u^2}{q}}\right|}\right)\xi_3, \end{aligned} \tag{3.49}$$

where

$$\varepsilon = \text{sign}(q), \quad \delta = P\text{sign}\left(p - \frac{u^2}{q}\right), \quad \lambda = \text{sign}\left(\frac{v^2}{\frac{u^2}{q} - p}\right)$$

we can transform the metric tensor (3.3) to the constant diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & -\lambda \end{pmatrix}. \tag{3.50}$$

3.5 Flat coordinates for the σ-model on $(5|1)$

The nonzero components of the affine connection for the metric (3.4) are

$$\begin{aligned} \Gamma^1_{11} &= -1 & \Gamma^2_{11} &= \frac{rp}{(ur - vg)}e^{y_1} & \Gamma^2_{12} &= \frac{gv}{(ur - vg)} \\ \Gamma^2_{13} &= \frac{vr}{(ur - vg)} & \Gamma^2_{22} &= \frac{g^2}{(ur - vg)}e^{-y_1} & \Gamma^2_{23} &= \frac{gr}{(ur - vg)}e^{-y_1} \\ \Gamma^2_{33} &= \frac{r^2}{(ur - vg)}e^{-y_1} & \Gamma^3_{11} &= -\frac{pg}{(ur - vg)}e^{y_1} & \Gamma^3_{21} &= -\frac{ug}{(ur - vg)} \\ \Gamma^3_{22} &= -\frac{g^3}{r(ur - vg)}e^{-y_1} & \Gamma^3_{31} &= -\frac{ur}{(ur - vg)} & \Gamma^3_{32} &= -\frac{g^2}{(ur - vg)}e^{-y_1} \\ \Gamma^3_{33} &= -\frac{gr}{(ur - vg)}e^{-y_1}. \end{aligned} \tag{3.51}$$

The general solution of the equations (3.13) is

$$\frac{1}{2}ap\, e^{y_1} + a\,(uy_2 + vy_3) + e^{-y_1}\left(\frac{(gy_2 + ry_3)(2cr + ary_3 + agy_2)}{2r} - b\right) + d, \tag{3.52}$$

where a, b, c, d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\begin{aligned}
\xi_1(y_1, y_2, y_3) &= -e^{-y_1} + d_1 \\
\xi_2(y_1, y_2, y_3) &= \frac{p\,r\cosh y_1 + r(uy_2 + vy_3) + e^{-y_1}\left(\frac{1}{2}(gy_2 + ry_3)^2 - v(gy_2 + ry_3)\right)}{ru - gv} + d_2 \\
\xi_3(y_1, y_2, y_3) &= \frac{-p\,g\cosh y_1 - g(uy_2 + vy_3) + e^{-y_1}\left(-\frac{g}{2r}(gy_2 + ry_3)^2 + u(gy_2 + ry_3)\right)}{ru - gv} + d_3
\end{aligned}$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} p & u & v \\ u & \frac{g^2}{r} & g \\ v & g & r \end{pmatrix}. \tag{3.53}$$

By the linear transformation

$$\begin{aligned}
y_1' &= \left(\sqrt{|p|}\right)\xi_1 + \left(\frac{u}{\varepsilon\sqrt{|p|}}\right)\xi_2 + \left(\frac{v}{\varepsilon\sqrt{|p|}}\right)\xi_3, \\
y_2' &= \left(\sqrt{\left|\frac{u^2}{p} - \frac{g^2}{r}\right|}\right)\xi_2 + \left(\frac{\delta\left(\frac{vu}{p} - g\right)}{\sqrt{\left|\frac{u^2}{p} - \frac{g^2}{r}\right|}}\right)\xi_3, \\
y_3' &= \left(\sqrt{\left|r - \frac{v^2}{p} + \frac{\left(\frac{vu}{p} - g\right)^2}{\left(\frac{u^2}{p} - \frac{g^2}{r}\right)}\right|}\right)\xi_3,
\end{aligned} \tag{3.54}$$

where

$$\varepsilon = \mathrm{sign}(p), \quad \delta = \mathrm{sign}\left(\frac{u^2}{p} - \frac{g^2}{r}\right), \quad \lambda = \mathrm{sign}\left(r - \frac{v^2}{p} + \frac{\left(\frac{vu}{p} - g\right)^2}{\frac{u^2}{p} - \frac{g^2}{r}}\right)$$

we can transform the metric tensor (3.4) to the constant diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{3.55}$$

3.6 Flat coordinates for the σ-model on $(7_0|1)$

The nonzero components of the affine connection for this metric are

$$\begin{array}{lll}
\Gamma^1_{23} = 1 & \Gamma^1_{32} = 1 & \Gamma^1_{33} = \frac{g}{p} - y_1 \\
\Gamma^2_{13} = -1 & \Gamma^2_{31} = -1 & \Gamma^2_{33} = -\frac{z}{p} - y_2.
\end{array} \tag{3.56}$$

The general solution of the equations (3.13) is

$$\xi(y_1, y_2, y_3) = \left(\frac{g}{p} - y_1\right)(iae^{iy_3} - ibe^{-iy_3}) - \left(\frac{z}{p} + y_2\right)(ae^{iy_3} + be^{-iy_3}) + cy_3 + d, \tag{3.57}$$

where a, b, c, d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\begin{aligned}
\xi_1(y_1,y_2,y_3) &= \left(-\frac{g}{p}+y_1\right)\cos(y_3)+\left(\frac{z}{p}+y_2\right)\sin(y_3)-\frac{z}{p}y_3+d_1\\
\xi_2(y_1,y_2,y_3) &= \left(\frac{g}{p}-y_1\right)\sin(y_3)+\left(\frac{z}{p}+y_2\right)\cos(y_3)-\frac{g}{p}y_3+d_2\\
\xi_3(y_1,y_2,y_3) &= y_3+d_3.
\end{aligned} \tag{3.58}$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} p & 0 & z \\ 0 & p & g \\ z & g & r \end{pmatrix}. \tag{3.59}$$

By the linear transformation

$$\begin{aligned}
\tilde{y}_1 &= \left(\sqrt{|p|}\right)\xi_1+\frac{\varepsilon z}{\sqrt{|p|}}\xi_3,\\
\tilde{y}_2 &= \left(\sqrt{|p|}\right)\xi_2+\frac{\varepsilon g}{\sqrt{|p|}}\xi_3,\\
\tilde{y}_3 &= \left(\sqrt{\left|r-\frac{z^2}{p}-\frac{g^2}{p}\right|}\right)\xi_3,
\end{aligned} \tag{3.60}$$

where

$$\varepsilon = \mathrm{sign}(p), \qquad \lambda = \mathrm{sign}\left(r-\frac{z^2}{p}-\frac{g^2}{p}\right)$$

we can transform the metric tensor (3.6) to the constant diagonal form

$$G'(\tilde{y}) = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{3.61}$$

3.7 Flat coordinates for the σ-model on $(1|6_0)$

The nonzero components of the affine connection for the metric (3.7) are

$$\begin{aligned}
\Gamma^1_{11} &= -\frac{(1+ky_1)k}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)} & \Gamma^1_{21} &= \frac{k^2y_2}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)}\\
\Gamma^1_{22} &= -\frac{(1+ky_1)k}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)} & \Gamma^2_{22} &= \frac{k^2y_2}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)}\\
\Gamma^2_{11} &= \frac{k^2y_2}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)} & \Gamma^2_{12} &= -\frac{(1+ky_1)k}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)}\\
\Gamma^3_{11} &= \frac{(1+ky_1)kqy_1}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)} & \Gamma^3_{21} &= -\frac{k^2qy_1y_2}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)}\\
\Gamma^3_{22} &= \frac{q(-1+k^2{y_2}^2-ky_1)}{(1+2ky_1+k^2{y_1}^2-k^2{y_2}^2)}.
\end{aligned} \tag{3.62}$$

The general solution of the equations (3.13) is

$$\xi(y_1,y_2,y_3) = \frac{(qa+4kb)}{4k^2}\ln\left(1+k(y_1-y_2)\right) + \frac{(qa+4kc)}{4k^2}\ln\left(1+k(y_1+y_2)\right)$$
$$-\frac{qa}{2k}y_1 + \frac{1}{4}\left(qa(y_1^2-y_2^2)\right) + ay_3 + d. \tag{3.63}$$

where a,b,c,d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\begin{aligned}
\xi_1(y_1,y_2,y_3) &= \frac{1}{2k}\ln(1+k(y_1-y_2)) + \frac{1}{2k}\ln(1+k(y_1+y_2)) + d_1\\
\xi_2(y_1,y_2,y_3) &= -\frac{1}{2k}\ln(1+k(y_1-y_2)) + \frac{1}{2k}\ln(1+k(y_1+y_2)) + d_2\\
\xi_3(y_1,y_2,y_3) &= \frac{q}{4k^2}\ln(1+k(y_1-y_2)) + \frac{q}{4k^2}\ln(1+k(y_1+y_2)) - \frac{q}{2k}y_1\\
&\quad +\frac{q}{4}({y_1}^2-{y_2}^2) + y_3 + d_3.
\end{aligned} \tag{3.64}$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} 0 & 0 & -k\\ 0 & q & 0\\ -k & 0 & 0\end{pmatrix}. \tag{3.65}$$

By the linear transformation

$$\begin{aligned}
y_1' &= \left(\frac{1}{2}\sqrt{|2k|}\right)\xi_1 + \left(\frac{1}{2}\sqrt{|2k|}\right)\xi_3,\\
y_2' &= (\sqrt{|q|})\xi_2,\\
y_3' &= \left(\frac{1}{2}\sqrt{|2k|}\right)\xi_1 - \left(\frac{1}{2}\sqrt{|2k|}\right)\xi_3,
\end{aligned} \tag{3.66}$$

where

$$\varepsilon = \mathrm{sign}(k), \qquad \delta = \mathrm{sign}(q)$$

we can transform the metric tensor (3.7) to the constant diagonal form

$$G'(y') = \begin{pmatrix} -\varepsilon & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & \varepsilon\end{pmatrix}. \tag{3.67}$$

3.8 Flat coordinates for the σ-model on $(5ii|6_0)$

The general solution of the equations (3.13) is

$$\begin{aligned}
\xi(y_1,y_2,y_3) &= -\frac{(2e^{y_2}qwa + (q(1+2w)a+2wb)y_2 + e^{-(y_1+y_2)}qa(2-w))}{4w^2}\\
&\quad -\frac{(e^{y_1+y_2}qw^2a)}{4w^2} + \frac{(qa(w^2-1)+4wc)\ln(1+(-1+e^{-(y_1+y_2)}))}{4w^2}\\
&\quad +\frac{(qa(1+2w)+2wb)\ln(-2w+e^{y_2}w+e^{-y_1}(1+w))}{4w^2} + ay_3\\
&\quad +\frac{(qwa+2wb)(y_1+y_2)}{4w^2} + \frac{e^{y_1}qa(1+w)}{2w} + \frac{qa(y_1+y_2)}{4w^2} + d
\end{aligned} \tag{3.68}$$

where a, b, c, d are integration constants. When we choose the initial conditions (3.14) then the flat coordinates in terms of the group coordinates are

$$\begin{aligned}
\xi_1(y_1, y_2, y_3) &= -\left(\frac{\ln(1 - w + e^{-(y1+y2)}w) + \ln(-2we^{y_1} + e^{y_1+y_2}w + 1 + w)}{2w}\right) + d_1 \\
\xi_2(y_1, y_2, y_3) &= \left(\frac{\ln(-2we^{y_1} + e^{y_1+y_2}w + 1 + w) - \ln(1 - w + e^{-(y_1+y_2)}w)}{2w}\right) + d_2 \\
\xi_3(y_1, y_2, y_3) &= -\left(\frac{2qe^{-y_2}w^2 + qwe^{-(y_1+y_2)} - 2qwe^{y_1} - qw^2e^{-(y_1+y_2)} + qwe^{y_1+y_2}}{4w^2}\right) \\
&\quad + \left(\frac{2e^{-y_2}qw^2 - qw^2 + e^{-(y_1+y_2)}q\ln(1 - w + e^{-(y_1+y_2)}w)}{4w^2}\right) e^{y1+y2} \\
&\quad + \frac{q}{4w^2}\ln(-2we^{-y_2} + w + e^{-(y_1+y_2)} + e^{-(y_1+y_2)}w) + \frac{q}{4w^2}(y_2 + y_1) \\
&\quad + y_3 + d_3
\end{aligned} \tag{3.69}$$

and

$$\widetilde{G}(\xi) = \begin{pmatrix} 0 & 0 & w \\ 0 & q & 0 \\ w & 0 & 0 \end{pmatrix}. \tag{3.70}$$

By the linear transformation

$$\begin{aligned}
y'_1 &= \frac{1}{2}\left(\sqrt{2\,|w|}\right)\xi_1 + \frac{1}{2}\left(\sqrt{2\,|w|}\right)\xi_3, \\
y'_2 &= (\sqrt{|q|})\xi_2, \\
y'_3 &= \frac{1}{2}\left(\sqrt{2\,|w|}\right)\xi_1 - \frac{1}{2}\left(\sqrt{2\,|w|}\right)\xi_3,
\end{aligned} \tag{3.71}$$

where

$$\varepsilon = \mathrm{sign}(w), \qquad \delta = \mathrm{sign}(q)$$

we can transform the metric tensor (3.9) to the constant diagonal form

$$G'(y') = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}. \tag{3.72}$$

4. Dilaton fields in flat and curved backgrounds

4.1 General dilatons in flat backgrounds

As mentioned in the section 3, the metrics (3.1)–(3.6) were obtained from the requirement that the vanishing β function equations are satisfied for the constant dilaton field. When we know the flat coordinates of these models we can easily find general forms of the dilaton fields that together with these metrics satisfy the vanishing β function equations (1.5)–(1.7).

In the flat coordinates y' these equations read

$$\frac{\partial^2\Phi'}{\partial y'_i \partial y'_j} = 0, \quad G'^{ij}\frac{\partial\Phi'}{\partial y'_i}\frac{\partial\Phi'}{\partial y'_j} = 0. \tag{4.1}$$

and from this form of the equations it is easy to see that the general form of the dilaton field for the flat metric $G_{ij}(y)$ is

$$\Phi(y) = c_1\xi_1(y) + c_2\xi_2(y) + c_3\xi_3(y) + c_0, \tag{4.2}$$

where $\xi_j(y)$ are coordinates that bring the flat metric to a constant form G'_{ij} and c_j are real constants satisfying

$$\sum_{j=1}^{3} G'^{ij} c_i c_j = 0. \tag{4.3}$$

For example, the general form of the dilaton field for the σ-model $(6_0|1)$ with the metric (3.5) that follow from (4.2) and (3.30) is

$$\begin{aligned}\Phi(y_1,y_2,y_3) = &\, c_1\left(\sqrt{|p|}\right)\left(y_1\cosh(y_3) + y_2\sinh(y_3) + \frac{v}{p}\sinh(y_3) - \frac{g}{p}\cosh(y_3)\right)\\ &+c_2\left(\sqrt{|p|}\right)\left(y_1\sinh(y_3) + y_2\cosh(y_3) + \frac{v}{p}\cosh(y_3) - \frac{g}{p}\sinh(y_3)\right)\\ &+c_3\left(\sqrt{\left|r + \left(\frac{g^2}{p} - \frac{v^2}{p}\right)\right|}\right) y_3 + c_0\end{aligned} \tag{4.4}$$

where $\mathrm{sign}(p)(c_1^2 - c_2^2) + \mathrm{sign}\left(r + \frac{g^2}{p} - \frac{v^2}{p}\right)c_3^2 = 0$.

By a similar way, i.e. as a linear combination of the flat coordinates, we can get the general dilaton fields for the σ-models with the metrics (3.1)–(3.4) and (3.6). If the metric is positively or negatively definite then the dilaton is constant.

We can also get dilaton fields more general than (3.8) and (3.10) for the models $(1|6_0)$ and $(5ii|6_0)$. The general form of the dilaton field for the σ-model $(1|6_0)$ is

$$\begin{aligned}\Phi(y) = &\, \frac{\sqrt{|2k|}}{4k}\left(c_1 + c_3 + (c_1 - c_3)\frac{q}{2k}\right)\ln|(1 + k(y_1 - y_2))(1 + k(y_1 + y_2))|\\ &+(c_1 - c_3)\left[\frac{1}{2}\sqrt{|2k|}\left(-\frac{q}{2k}y_1 + \frac{q}{4}({y_1}^2 - {y_2}^2) + y_3\right)\right]\\ &+c_2\frac{\sqrt{|q|}}{2k}\ln\left|\frac{1 + k(y_1 + y_2)}{1 + k(y_1 - y_2)}\right| + c_0\end{aligned} \tag{4.5}$$

where $\mathrm{sign}(q)c_2^2 + \mathrm{sign}(k)(c_3^2 - c_1^2) = 0$. For special choice of constants $c_1 = c_3 = 2k/\sqrt{|2k|}$, $c_2 = 0$, we get the dilaton field (3.8) obtained in [2] by the Poisson-Lie T-duality. The general form of the dilaton field for the σ-model $(5ii|6_0)$ can be obtained from (3.71) as well but it is rather extensive to display.

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By the Poisson-Lie transformation of (4.2) we can get dilatons for the σ-models dual to the flat ones but, as mentioned before, only if the necessary conditions are satisfied. Due to (2.6) and (4.2) the condition (2.9) reads

$$R^{jk}\left(c_m\frac{\partial\xi_m}{\partial y_j}-\frac{\partial}{\partial y_j}\ln|\,\mathrm{Det}[\,a(g)\,(\mathbf{1}+\Pi(g)E_0)\,]\,|\right)|_{y=0}=0. \tag{4.6}$$

Moreover, the matrix $\Pi(g)$ vanishes for the semiabelian Manin triples and the flat coordinates can be chosen to satisfy $\frac{\partial\xi_m}{\partial y_j}(0)=\delta_{mj}$. The condition (4.6) then simplifies to

$$R^{jk}\left(c_j-\frac{\partial}{\partial y_j}\ln|\,\mathrm{Det}\,a(g)\,|\right)|_{y=0}=0. \tag{4.7}$$

4.2 Dilatons for σ-models in curved background dual to (5|1)

The first σ-model in the curved background that we are going to investigate is given by the metric

$$\widetilde{G}_{ij}(u)=\begin{pmatrix} e^{-2\epsilon u_3}Q & \epsilon\, e^{-2\epsilon u_3}Q & V\cosh u_3-H\sinh u_3\\ \epsilon\, e^{-2\epsilon u_3}Q & e^{-2\epsilon u_3}Q & H\cosh u_3-V\sinh u_3\\ V\cosh u_3-H\sinh u_3 & H\cosh u_3-V\sinh u_3 & J\end{pmatrix}, \tag{4.8}$$

where $\epsilon=\pm1$ and Q,V,H,J are constants. This metric has nonvanishing Ricci tensor but its Gauss curvature is zero. It belongs to the σ-model corresponding to the $(6_0|1)$ decomposition of the $DD11$ (for the notation see [7]) and $\widetilde{E}_0=\widetilde{G}(0)$. On the other hand, it can be obtained by the Poisson-Lie transformation (2.2), (2.3) from the metric

$$G_{ij}(y)=\begin{pmatrix} 0 & 0 & v\,\mathrm{e}^{-y_1}\\ 0 & q\,\mathrm{e}^{-2y_1} & 0\\ v\,\mathrm{e}^{-y_1} & 0 & 0\end{pmatrix}, \tag{4.9}$$

where q,v are constants. The latter metric is flat and corresponds to the (5|1) decomposition of the $DD11$ and $E_0=G(0)$.

The matrix (2.1) that transform the Manin triple (5|1) to $(6_0|1)$ and the metric (4.9) to (4.8) is

$$\begin{pmatrix} P & T\\ R & S\end{pmatrix}=\begin{pmatrix} -\beta+\frac{1}{2}\alpha & \epsilon(\beta+\frac{1}{2}\alpha) & -\epsilon & 0 & 0 & 0\\ 0 & 0 & 0 & \epsilon & 1 & \alpha\\ -\epsilon & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\epsilon\\ \frac{1}{2}\epsilon & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\epsilon & \frac{1}{2} & \beta\end{pmatrix}, \tag{4.10}$$

where relations between the constants are

$$q=Q^{-1},\qquad v=V-\epsilon\,H,\qquad \alpha=\frac{\epsilon\,V+H}{2\,Q},\qquad \beta=\epsilon\frac{\alpha^2Q-J}{2v}. \tag{4.11}$$

In fact, the metric (4.8) is the most general that can be obtained by the Poisson-Lie transformation from a flat metric corresponding to the (5|1) decomposition of the $DD11$.

General form of the dilaton field for the metric (4.9) is given by (4.2) where

$$\begin{aligned}\xi_1(y_1,y_2,y_3) &= -e^{-y_1}\\ \xi_2(y_1,y_2,y_3) &= e^{-y_1}y_2\\ \xi_3(y_1,y_2,y_3) &= \frac{q}{2v}e^{-y_1}{y_2}^2+y_3.\end{aligned} \tag{4.12}$$

These are the coordinates that bring the flat metric to its constant form $G'_{ij}=G_{ij}(0)$.

The formula (2.4) for the general dilaton of the σ-model given by (4.8) yields

$$\Phi_U(y) = -2y_1 - c_1e^{-y_1} + c_2e^{-y_1}y_2 + c_3\left(\frac{q}{2v}e^{-y_1}{y_2}^2+y_3\right)+c_0, \tag{4.13}$$

where the coefficients satisfy the equation (4.3) that in this case reads

$$v\,c_2^2 + 2q\,c_1c_3 = 0. \tag{4.14}$$

However, this is not yet the final form of the dilaton field because it is expressed in terms of the coordinates y of the σ-model given by (4.9) and it must be transformed to the coordinates u of the σ-model given by (4.8). The transformation formulas between these coordinates follow from two different decompositions of elements of the Drinfel'd double $DD11$, namely from the relation

$$e^{-y_1X_1}e^{-y_2X_2}e^{-y_3X_3}e^{-\tilde{y}_1\tilde{X}_1}e^{-\tilde{y}_2\tilde{X}_2}e^{-\tilde{y}_3\tilde{X}_3} = e^{-u_3U_3}e^{-u_2U_2}e^{-u_1U_1}e^{-\tilde{u}_1\tilde{U}_1}e^{-\tilde{u}_2\tilde{U}_2}e^{-\tilde{u}_3\tilde{U}_3}, \tag{4.15}$$

where $X_j,\widetilde{X}_j$ are generators corresponding to the decomposition $(5|1)$ of the Drinfel'd double $DD11$ and $U_j,\widetilde{U}_j$ are generators of the decomposition $(6_0|1)$. They can be related by (4.10). Coordinates y in terms of u are then expressed as

$$\begin{aligned}y_1 &= -\epsilon\,u_3,\\ y_2 &= \frac{\epsilon\,\widetilde{u}_1+\widetilde{u}_2}{2},\\ y_3 &= \frac{-\epsilon\,u_1+u_2}{2}+\beta u_3,\\ \widetilde{y}_1 &= \beta(-\widetilde{u}_1+\epsilon\,\widetilde{u}_2)-\epsilon\,\widetilde{u}_3+\frac{1}{2}(\widetilde{u}_1+\epsilon\,\widetilde{u}_2)(\alpha+u_1+\epsilon\,u_2+\epsilon\,\alpha u_3),\\ \widetilde{y}_2 &= \epsilon\,u_1+u_2+\alpha\,u_3,\\ \widetilde{y}_3 &= -\epsilon\,\widetilde{u}_1+\widetilde{u}_2.\end{aligned} \tag{4.16}$$

We can see that unless $c_2=0$, $c_3=0$ the dilaton (4.13) depends on the coordinate $\epsilon\,\widetilde{u}_1+\widetilde{u}_2$. It is not admissible and thus the general form of dilaton obtained by the Poisson-Lie transformation for the metric (4.8) is

$$\widetilde{\Phi}(u) = \Phi_U(y(u)) = 2\epsilon u_3 + c_1e^{\epsilon u_3}+c_0. \tag{4.17}$$

We have checked that the vanishing β function equations for $\widetilde{\Phi}(u)$ and $\widetilde{G}_{ij}(u)$ given by (4.8) are satisfied.

Note that the condition $c_3 = 0, c_2 = 0$ is more strict than the necessary condition (4.7) that implies $c_2 = 0$ only. It means that the necessary condition (2.7) is not sufficient for the Poisson-Lie transformation of the dilaton.

By other plurality transformations of (4.9) we can get σ-models with curved background corresponding to the decompositions $(1|6_0)$ and $(5.ii|6_0)$ of the DD11. For the dilaton fields the formula (2.4) could be again used but we were not able to express the coordinates $y, \tilde{y}$ in terms of $u, \tilde{u}$ from the relation (4.15) in these cases.

4.3 Dilatons for σ-models in curved background dual to $(4|1)$

A bit more complicated σ-model is given by the metric $\widetilde{G}_{ij}(u)$, where

$$\begin{aligned}
\widetilde{G}_{11}(u) &= \widetilde{G}_{22}(u) = e^{-2\epsilon u_3} Q \\
\widetilde{G}_{12}(u) &= \widetilde{G}_{21}(u) = \epsilon\, e^{-2\epsilon u_3} Q \\
\widetilde{G}_{13}(u) &= \widetilde{G}_{31}(u) = V \cosh u_3 - H \sinh u_3 \\
\widetilde{G}_{23}(u) &= \widetilde{G}_{32}(u) = H \cosh u_3 - V \sinh u_3 \\
\widetilde{G}_{33}(u) &= J - Q\,(V - \epsilon\, H)^2 u_3^2
\end{aligned} \tag{4.18}$$

where $\epsilon = \pm 1$ and Q, V, H, J are constants. Again, this metric has nonvanishing Ricci tensor and its Gauss curvature is zero. It belongs to the σ-model corresponding to the $(6_0|2)$ decomposition of the $DD12$ and $\widetilde{E}_0 = \widetilde{G}(0)$. Besides that it can be obtained by the Poisson-Lie transformation (2.2), (2.3) from the metric

$$G_{ij}(y) = \begin{pmatrix} 0 & v\,\mathrm{e}^{-y_1} y_1 & v\,\mathrm{e}^{-y_1} \\ v\,\mathrm{e}^{-y_1} y_1 & q\,\mathrm{e}^{-2y_1} & 0 \\ v\,\mathrm{e}^{-y_1} & 0 & 0 \end{pmatrix}, \tag{4.19}$$

where q, v are constants. This metric is flat and corresponds to the $(4|1)$ decomposition of the $DD12$.

The matrix (2.1) that transform the metric (4.19) to (4.18) is

$$\begin{pmatrix} P & T \\ R & S \end{pmatrix} = \begin{pmatrix} -\beta - \frac{1}{2}\alpha & \epsilon(\beta - \frac{1}{2}\alpha) & -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & -1 & \alpha \\ -\epsilon & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon \\ -\frac{1}{2}\epsilon & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\epsilon & \frac{1}{2} & \beta \end{pmatrix}, \tag{4.20}$$

where the relations between the constants are

$$q = Q^{-1}, \quad v = V - \epsilon\, H, \quad \alpha = -\frac{\epsilon\, V + H}{2\,Q}, \quad \beta = \epsilon \frac{\alpha^2 Q - J}{2v}. \tag{4.21}$$

The general dilaton field for the σ-model with the flat metric (4.19), which is a special case of the metric (3.3), is obtained by insertion of the flat coordinates (3.47) into (4.2)

$$\begin{aligned}
\Phi_U(y) = & -2y_1 + c_0 - c_1 e^{-y_1} + c_2 \left(\frac{v}{q} \left(y_1 + e^{-y_1} \right) + e^{-y_1} y_2 \right) + \\
& c_3 \left(\frac{v}{q} (y_1 - \sinh y_1) + \frac{q}{2v} e^{-y_1} y_2^2 + \left(e^{-y_1} + y_1 - 1 \right) y_2 + y_3 \right)
\end{aligned} \tag{4.22}$$

and the constants c_j satisfy

$$v\,c_2^2 + 2q\,c_1c_3 = 0.$$

The formula (2.4) for the dilaton of the σ-model with the the curved metric (4.18) then yelds $\Phi_U = -2y_1 + \Phi(y)$ but to get the final form of the dilaton field we must transform it to the coordinates u. The transformation formulas follow from decompositions of elements of the Drinfel'd double $DD12$, namely from the relation (4.15) where $X_j, \widetilde{X}_j$ are generators corresponding to the decomposition $(4|1)$ and $U_j, \widetilde{U}_j$, related by (2.1) and (4.20), correspond to the decomposition $(6_0|2)$. Coordinates y in terms of u are then expressed as

$$\begin{aligned}
y_1 &= -\epsilon\,u_3,\\
y_2 &= -\frac{\epsilon\,\widetilde{u}_1 + \widetilde{u}_2}{2},\\
y_3 &= \frac{-\epsilon\,u_1 + u_2}{2} + \beta u_3,\\
\widetilde{y}_1 &= \beta(-\widetilde{u}_1 + \epsilon\widetilde{u}_2) + \frac{1}{2}(\widetilde{u}_1 + \epsilon\widetilde{u}_2)(-\alpha + u_1 + \epsilon\,u_2 - \epsilon\,\alpha\,u_3) - \frac{1}{4}\widetilde{u}_1^2 + \frac{1}{2}\epsilon\,\widetilde{u}_1\widetilde{u}_2 + \frac{1}{4}\widetilde{u}_2^2 - \epsilon\,\widetilde{u}_3,\\
\widetilde{y}_2 &= -\epsilon\,u_1 - u_2 + \alpha\,u_3,\\
\widetilde{y}_3 &= -\epsilon\,\widetilde{u}_1 + \widetilde{u}_2.
\end{aligned} \tag{4.23}$$

In order that the dilaton does not depend on the coordinate $\epsilon\,\widetilde{u}_1 + \widetilde{u}_2 = y_2$ we must set $c_2 = 0$, $c_3 = 0$ and the general form of the dilaton obtained by the Poisson-Lie transformation for the metric (4.18) is

$$\widetilde{\Phi}(u) = \Phi_U(y(u)) = 2\epsilon u_3 + c_1 e^{\epsilon u_3} + c_0. \tag{4.24}$$

The vanishing β function equations are satisfied.

4.4 Dilatons for σ-models in a flat background dual to $(4|1)$

In the previous subsections we have used the general dilatons of flat models for producing dilatons of the dual curved models. An interesting and important question is whether the general dilatons enable constructing dual dilatons in cases where the constant dilatons fail. More precisely, are there examples where the constant dilaton cannot produce a dual dilaton independent of the auxiliary variables but a more general one can?

We shall show that there are examples where the constant dilatons do not satisfy the necessary condition (2.5) for the Poisson-Lie transformation but a more general do. Unfortunately, this does not give definite answer to the question because for construction of the dual dilaton we have to find the transformation of coordinates given by (4.15) which may be a difficult problem.

An example of the flat σ-model for which the trivial dilaton is not dualizable is given by the metric

$$\widetilde{G}_{ij}(u) = \Delta(u)\begin{pmatrix} Q - 4QV^2u_1^2 & V u_1 + Q\left(4V^2u_1u_2 - 1\right) & V \\ V u_1 + Q\left(4V^2u_1u_2 - 1\right) & -4QV^2u_2^2 + Q + 2Vu_1 & V \\ V & V & 0 \end{pmatrix}, \tag{4.25}$$

where

$$\Delta(u) = (1 - V^2(u_1 + u_2)^2).$$

It corresponds to the decomposition $(2|6_0)$ of the Drinfel'd double $DD12$ and is related to the metric (4.19) by the Poisson-Lie transformation (2.2), (2.3) where

$$\begin{pmatrix} P & T \\ R & S \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}, \quad q = 4Q, \ v = 1/V. \tag{4.26}$$

Both the metrics (4.19) and (4.25) are flat so that their general dilaton fields can be obtained by insertion of the flat coordinates into (4.2). By this way we get dilaton (4.22) and

$$\begin{aligned} \widetilde{\Phi}(u) = \ & \widetilde{c}_0 + \widetilde{c}_1 \operatorname{arctanh}(V(u_1+u_2)) + \\ & \widetilde{c}_2 \left[\ln|1 - V^2(u_1+u_2)^2| + 4QV(u_1-u_2)\right] + \\ & \widetilde{c}_3 \Bigg[8Q\, u_3 - \frac{1}{V^2}\operatorname{arctanh}(V(u_1+u_2)) + \\ & 2Q(u_1+u_2)\left(6u_1 - 2u_2 + \frac{1}{QV}\right) - 4Q(u_1-u_2)^2(1 + 8QV(u_1+u_2))\Bigg], \end{aligned} \tag{4.27}$$

where the coefficients satisfy the equation

$$\widetilde{c}_1\widetilde{c}_3 + \widetilde{c}_2^2\, V^2 = 0. \tag{4.28}$$

It means that we have two flat metrics related by the Poisson-Lie transformation, we know their general dilatons, and we can ask if at least some parts of the dilatons can be transformed one to the other.

The necessary condition (4.7) for the dilaton transformation given by the matrix (4.26) *is not satisfied by the constant dilaton but only by*

$$\Phi(y) = c_0 - 2\, e^{-y_1} \tag{4.29}$$

obtained from (4.22) by setting $c_1 = 2, c_2 = 0, c_3 = 0$. The formula (2.4) for the dual dilaton corresponding to the metric (4.25) then gives

$$\Phi_U = c_0 - 2\, e^{-y_1} - 2y_1 + \ln|1 - V^2(u_1+u_2)^2|. \tag{4.30}$$

We can see by comparing the number of free constants in (4.27) and (4.30) that the Poisson-Lie transformation can produce only a special form of the dilaton for the dual model. The problem however is that we do not know if y_1 depends on $\widetilde{u}_j$ or not because we are not able to solve the relation (4.15) in this case.

A similar situation, namely that the constant dilaton does not satisfy the necessary condition for construction of the dual dilaton but a more general does, happens in the models following from the pluralities $(4|1) \cong (4ii|6_0)$ and $(5|1) \cong (5ii|6_0) \cong (1|6_0)$. Unfortunately, once again in these case we are not able to solve the relation (4.15).

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5. Conclusions

The Poisson-Lie T-duality does not preserve the geometric properties of the backgrounds so that it can relate σ-models in curved and flat backgrounds. We have used this fact to solve the vanishing β function equations (1.5)–(1.7) in curved backgrounds applying the Poisson-Lie transformation (2.4) to general dilatons for the flat metrics.

The equations for the dilaton field of the flat σ-model are easily solvable in the flat coordinates. To get the general dilatons for the flat metrics in terms of the group coordinates we need the transformations between the group and flat coordinates of the σ-models. These transformations were found for three-dimensional flat σ-models and their explicit forms were presented in the Section 3. The transformations can be used for many other purposes. The equations of motion of the σ-models with the flat metrics are easily solvable in terms of the flat coordinates. If the investigated models are Poisson-Lie T-dual or plural to σ-models with nontrivial backgrounds then the transformation to the group coordinates offers a possibility to find classical solutions in the nontrivial background. An example of such solution was given in [8] and other models are being solved now.

In the section 4, new dilaton field for the metrics (4.8) and (4.18), both having nontrivial Ricci tensor, was found. It is the most general dilaton that can be obtained by the Poisson-Lie transformation from the general dilatons (4.13), (4.22) of the dual flat metrics (4.9) and (4.19). These cases show that the necessary condition (2.7) for the applicability of the formula (2.4) is not sufficient. An interesting but yet unsolved question is whether the dilaton (4.17) obtained by the Poisson-Lie transformation is the general solution of the vanishing β function equations for the curved backgrounds (4.8) and (4.18). The results of transformations of dual flat models indicate that this need not be so.

In the subsection 4.4 we have tried to answer the question whether the general dilatons enable to satisfy the conditions (2.5) and (2.9) for construction of dual dilatons in cases where the constant dilatons fail. We were able to find examples for which the necessary condition (2.9) is satisfied only for a nonconstant dilaton, nevertheless, it is not clear whether the nonconstant dilaton can be transformed to a dual one. The reason is that in these cases we are not currently able to determine the dependence of the relevant y coordinates on the variables u, $\widetilde{u}$ implicitly given by (4.15). On the other hand, there are cases when only the constant dilaton may be inserted into the formula (2.4) because all y coordinates depend on the inadmissible auxiliary variables $\widetilde{u}$. Examples of this are models corresponding to the dual Manin triples $(1|2) \cong (2|1)$, $(1|7_0) \cong (7_0|1)$ whose dilatons were published in [2].

The examples investigated in this paper show that there are many dual models with dilatons that cannot be related by the formula (2.4). That might indicate that a more general prescription for the dilaton transformation may exist.

Acknowledgments

This work was supported by the project of the Grant Agency of the Czech Republic No. 202061480 and by the research plan LC527 15397/2005–31 of the Ministry of Education of the Czech Republic. Useful comments of Josef Mikeš and Libor Šnobl are gratefully acknowledged.

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